

TRANSFORMATION
OF
LINEAR PARTIAL DIFFERENTIAL EQUATIONS

BY
HUNG CHI CHANG

張 鴻 基

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy in the University of Michigan

*(Reprinted from the Transactions of the Science Society
of China, Vol. VII, No. 2, 1931.)*

Ann Arbor
1929



TRANSFORMATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

CHAPTER I

ON A CERTAIN CLASS OF NON-ANALYTIC FUNCTIONS WHICH CAN BE TRANSFORMED INTO ANALYTIC FUNCTIONS OF A NEW VARIABLE. †

In Professor K. P. Williams' paper* on "A generalization of Cauchy-Riemann equations" he says that a function $u+iv$ can be considered as an analytic function of $z=Ay$ if u and v satisfy the system of equations

$$(1) \quad \begin{cases} Au_x + u_y + v_x = 0 \\ u_x - Av_x - v_y = 0 \end{cases}$$

In this chapter we shall treat a still more general case where a given function $u+iv$ may be reduced to an analytic function when u and v satisfy the following system of equations

$$(2) \quad \begin{cases} Au_x + Bu_y + Cv_x + Dv_y = 0 \\ A'u_x + B'u_y + C'v_x + D'v_y = 0 \end{cases}$$

with the condition

$$4(BC-AD)(B'C'-A'D')-(AD'+A'D-BC'-B'C)^2 > 0$$

on the coefficients of (2) and when the partial derivatives of u and v are continuous. This has been accomplished by applying linear transformations to the function and to the variable.

1. Let the given function $u+iv$ which satisfies the system of equations (2) be transformed into another function $u'+iv'$ which will be analytic in the new variable $z'=x'+iy'$ where

† *Presented at the Conference of the Science Society of China at Tsingtao, Aug. 1930, also offered as a dissertation in the Univ. of Michigan
* and Read before the American Mathematical Society at Cincinnati, Ohio, Nov. 30 to Dec. 1, 1928.

$$(3) \quad \begin{cases} u' = au + bv \\ v' = cu + dv \end{cases} \quad (4) \quad \begin{cases} u = a'u' + b'v' \\ v = c'u' + d'v' \end{cases}$$

$$(5) \quad \begin{cases} x = \alpha x' + \beta y' \\ y = \gamma x' + \delta y' \end{cases} \quad (6) \quad \begin{cases} x' = \alpha'x + \beta'y \\ y' = \gamma'x + \delta'y \end{cases}$$

From the definition of an analytic function we have the Cauchy-Riemann equations

$$(7) \quad \frac{\partial u'}{\partial x'} = \frac{\partial v'}{\partial y'}; \quad \frac{\partial u'}{\partial y'} = -\frac{\partial v'}{\partial x'}$$

which, by partial differentiation under change of variable and function according to (3) and (5), can be reduced readily to

$$(8) \quad \begin{cases} u_x (a\alpha - c\beta) + u_y (a\gamma - c\delta) + v_x (b\alpha - d\beta) + v_y (b\gamma - d\delta) = 0 \\ u_x (a\beta + c\alpha) + u_y (a\delta + c\gamma) + v_x (b\beta + d\alpha) + v_y (b\delta + d\gamma) = 0 \end{cases}$$

And, of course, (8) can be brought back to (7) through the inverse transformations (4) and (6).

Now, consider a system of equations equivalent to system (2), namely:

$$(9) \quad \begin{cases} u_x (\lambda A + \mu A') + u_y (\lambda B + \mu B') + v_x (\lambda C + \mu C') + v_y (\lambda D + \mu D') = 0 \\ u_x (\varrho A + \sigma A') + u_y (\varrho B + \sigma B') + v_x (\varrho C + \sigma C') + v_y (\varrho D + \sigma D') = 0 \end{cases}$$

where $\lambda\sigma - \mu\varrho \neq 0$. Identifying the coefficients of (9) with those of (8), which consequently makes (9) transformable into (7), we obtain 8 equations involving 12 introduce unknown constants, $a, b, c, d, \alpha, \beta, \gamma, \delta, \lambda, \mu, \varrho$ and σ

$$(10) \quad \left\{ \begin{array}{l} a\alpha - c\beta = \lambda A + \mu A' \equiv a_1 \\ a\gamma - c\delta = \lambda B + \mu B' \equiv a_2 \\ a\alpha - d\beta = \lambda C + \mu C' \equiv a_3 \\ b\gamma - d\delta = \lambda D + \mu D' \equiv a_4 \\ a\beta + c\alpha = \varrho A + \sigma A' \equiv b_1 \\ a\delta + c\gamma = \varrho B + \sigma B' \equiv b_2 \\ b\beta + d\alpha = \varrho C + \sigma C' \equiv b_3 \\ b\delta + d\gamma = \varrho D + \sigma D' \equiv b_4 \end{array} \right.$$

where a_j and b_j ($j=1, 2, 3, 4$) are used only as abbreviations for future convenience.

2. Equations (10) can be solved for a , b , c , and d as follows

$$(11) \quad \begin{cases} a = \frac{\beta a_2 - \delta a_1}{\beta \gamma - \alpha \delta} = \frac{\gamma b_1 - \alpha b_2}{\beta \gamma - \alpha \delta} \\ b = \frac{\beta a_4 - \delta a_3}{\beta \gamma - \alpha \delta} = \frac{\gamma b_3 - \alpha b_4}{\beta \gamma - \alpha \delta} \\ c = \frac{\alpha a_2 - \gamma a_1}{\beta \gamma - \alpha \delta} = \frac{\beta b_2 - \delta b_1}{\beta \gamma - \alpha \delta} \\ d = \frac{\alpha a_4 - \gamma a_3}{\beta \gamma - \alpha \delta} = \frac{\beta b_4 - \delta b_3}{\beta \gamma - \alpha \delta} \end{cases}$$

provided that $\beta \gamma - \alpha \delta \neq 0$.

For the existence of α , β , γ and δ (not all zero) satisfying equations (11), considering a , b , c and d eliminated, it is necessary and sufficient that

$$(12) \quad \Delta \equiv \begin{vmatrix} b_2 & a_2 & -b_1 & -a_1 \\ b_4 & a_4 & -b_3 & -a_3 \\ a_2 & -b_2 & -a_1 & b_1 \\ a_4 & -b_4 & -a_3 & b_3 \end{vmatrix} = 0$$

Evaluating the determinant in (12) by Laplace's development according to two-rowed minors from the top two rows we get

$$(13) \quad \begin{aligned} -\Delta &= (a_3 b_2 - a_1 b_4)^2 + (b_2 b_3 - b_1 b_4)^2 + (a_2 a_3 - a_1 a_4)^2 \\ &+ (a_2 b_3 - a_4 b_1)^2 - 2a_3 a_4 b_1 b_2 - 2a_1 a_2 b_3 b_4 + 2a_1 a_4 b_2 b_3 + 2a_2 a_3 b_1 b_4 \\ &= \left[(b_2 b_3 - b_1 b_4) - (a_2 a_3 - a_1 a_4) \right]^2 \\ &+ \left[(a_3 b_2 - a_1 b_4) + (a_2 b_3 - a_4 b_1) \right]^2 = 0 \end{aligned}$$

Therefore,

$$(14) \quad b_2 b_3 - b_1 b_4 = a_2 a_3 - a_1 a_4$$

and

$$(15) \quad a_3 b_2 - a_1 b_4 = a_4 b_1 - a_2 b_3$$

Substituting values of a_i and b_i as defined by (10) in (14) and (15) we get

$$(16) \quad (BC-AD)(\varrho^2-\lambda^2)+(BC'+B'C-AD'-A'D)(\varrho\sigma-\lambda\mu) \\ + (B'C'-A'D')(\sigma^2-\mu^2)=0$$

$$(17) \quad 2\lambda\varrho(BC-AD)+2\mu\sigma(B'C'-A'D') \\ + (BC'+B'C-AD'-A'D)(\lambda\sigma+\mu\varrho)=0$$

Equations (16) and (17) can further be simplified by combining them thus,

$$(18) \quad BC'+B'C-AD'-A'D = \frac{-2\lambda\varrho(BC-AD)-2\mu\sigma(B'C'-A'D')}{\lambda\sigma-\mu\varrho} \\ = \frac{-(BC-AD)(\varrho^2-\lambda^2)-(B'C'-A'D')(\sigma^2-\mu^2)}{\varrho\sigma-\mu\lambda}$$

From the second equation in (18) we may collect terms and write

$$(19) \quad \frac{BC-AD}{B'C'-A'D'} = \frac{\sigma^2+\mu^2}{\lambda^2+\varrho^2}$$

In like manner the first part of (18) takes the following form

$$(20) \quad \frac{AD'+A'D-BC'-B'C}{B'C'-A'D'} = 2 \frac{(\lambda\mu+\varrho\sigma)}{\lambda^2+\varrho^2}$$

Therefore, a necessary condition is

$$(21) \quad \frac{BC-AD}{B'C'-A'D'} > 0 \quad \text{or, } (BC-AD)(B'C'-A'D') > 0$$

because the right-hand member of (19) is always positive. If (21) is satisfied then (19) can be solved for λ , μ , ϱ and σ . In fact, put

$$(22) \quad \begin{aligned} \lambda &= \sqrt{B'C'-A'D'} \sin \theta \\ \varrho &= \sqrt{B'C'-A'D'} \cos \theta \\ \sigma &= 0 \\ \mu &= \sqrt{BC-AD} \end{aligned}$$

where θ is arbitrary, then (19) is satisfied and equation (20) will be satisfied if we have

$$(23) \quad \sin \theta = \frac{AD' + A'D - BC' - B'C}{2\sqrt{B'C' - A'D'} \sqrt{BC - AD}}$$

In (23) a θ can be found when, and only when, the righthand member is less than one in absolute value, that is,

$$(24) \quad 4(B'C' - A'D')(BC - AD) - (AD' + A'D - BC' - B'C)^2 > 0$$

which evidently implies (21). Therefore, (24) is a necessary and sufficient condition for (19) and (20) to be solvable in λ , μ , ρ and σ . If (19) and (20) are thus satisfied then (18) must be satisfied since

$$(\rho \sigma - \lambda \mu)(\lambda \sigma + \mu \rho) \neq 0$$

is assured by condition (24) or, more directly, by (22) and (23).

Tracing back through (14) and (15) condition (12) is found to be fulfilled. So α , β , γ and δ can be solved for, say, as cofactors of b_2 , a_2 , b_1 and a_1 respectively from the determinant in (12). As soon as α , β , γ and δ are found and if $\beta \gamma - \alpha \delta \neq 0$ then values of a , b , c and d are given by (11). But $\beta \gamma - \alpha \delta$ cannot be zero for, if it were, equation (10) can be written

$$(25) \quad \begin{cases} \xi(\lambda A + \mu A') = \lambda B + \mu B' \\ \xi(\lambda C + \mu C') = \lambda D + \mu D' \\ \xi(\rho A + \sigma A') = \rho B + \sigma B' \\ \xi(\rho C + \sigma C') = \rho D + \sigma D' \end{cases}$$

where ξ is an arbitrary constant. Since σ was chosen to be zero (25) can be reduced to

$$(26) \quad \xi A = B, \quad \xi C = D, \quad \xi A' = B', \quad \xi C' = D'$$

Consequently, $BC - AD = 0$ and $B'C' - A'D' = 0$

which contradicts condition (24). Therefore, $\beta \gamma - \alpha \delta \neq 0$.

In similar fashion it can be shown that $ad - bc \neq 0$. Thus, the theorem reads: *Given a function $u + iv$ satisfying equations (2), a necessary and sufficient condition for the possibility of trans-*

forming it into another function $u' + iv'$ which will satisfy the Cauchy-Riemann equations (7) is that (24) holds.

3. If we further assume that the partial derivatives of u and v with respect to x and y are continuous, then, the partial derivatives of u' and v' with respect to x' and y' are also continuous; and this, together with the Cauchy-Riemann equations (7), is sufficient for the function $u' + iv'$ to be analytic in $z' = x' + iy'$. Therefore, $u' + iv'$ is expressible as a power series in z' or, what is the same thing, in x' and y' . Translating these results into the language of the given function we have the following

Corollary. Any function $u + iv$ which satisfies (2) under condition (24) and whose partial derivatives are continuous can be developed in a power series in x and y .

4. It is also interesting to note that the second order differential equation derived from (2) by eliminating u or v , namely

$$(27) \quad f_{xx} AC' - A'C + f_{xy} (AD' - A'D + BC' - B'C) + f_{yy} (BD' - B'D) = 0$$

can be reduced to a Laplace's equation because the discriminant of (27) is

$$(28) \quad \Delta = \frac{1}{4} (AD' - A'D + BC' - B'C)^2 - (AC' - A'C) (BD' - B'D) \\ = \frac{1}{4} (AD' + A'D - BC' - B'C)^2 - (B'C' - A'D') (BC - AD)$$

which is always negative by (24)—the condition required for a linear homogeneous second order differential equation to be reducible to a Laplace's equation*.

*Goursat-Hedrick, Mathematical Analysis, Vol. I, pp. 72-73.

CHAPTER II

GENERALIZATION OF NEWTONIAN EQUATIONS

1. In Chapter I we found an extended class of analytic function through generalizing Cauchy-Riemann equations. However, Cauchy-Riemann equations involve only two functions and two variables and, therefore, do not apply in the case of three dimensions (three functions and three variables). In this chapter we shall take up a "set of equations" involving three functions and three variables, which plays the same role in three dimensions as Cauchy-Riemann equations do in two dimensions. It is that "set of equations" that we shall try to generalize by means of transformations as we did in Chapter I.

From Newton's laws of motion and gravitation in Mechanics we know

$$(1.1)^* \quad u'_{y'} - v'_{x'} = 0$$

$$(1.2) \quad u'_{z'} - w'_{x'} = 0$$

$$(1.3) \quad v'_{z'} - w'_{y'} = 0$$

$$(1.4) \quad u'_{x'} + v'_{y'} + w'_{z'} = 0$$

which we shall call Newtonian equations†. When w' is set equal to zero in (1) then Cauchy-Riemann equations are obtained there from and consequently are a special case of the Newtonian equations. But our problem is not to study analytic functions in three dimensions; instead, it is rather to find a generalization of the Newtonian equations. In short, our problem reads: What equations can be transformed into Newtonian equations?

*Hereafter we shall speak of "equations (1)" meaning equations (1.1), (1.2), (1.3) and (1.4). The same rule of notation holds for other sets of equations.

†Equations (1.1), (1.2) and (1.3) are conditions for the vector $ui + vj + wk$ to be what we call lamellar vector in Vector Analysis and (1.4) the condition for solenoidal vector. In the future we shall refer to the former as "rotation" equations and the latter as "divergence" equation.

Suppose we are given with four linear homogeneous differential equations involving three functions and three variables in the most general form:

$$(2.1) \quad B_{11}u_x + B_{12}u_y + B_{13}u_z + B_{21}v_x + B_{22}v_y + B_{23}v_z \\ + B_{31}w_x + B_{32}w_y + B_{33}w_z = 0$$

$$(2.2) \quad C_{11}u_x + C_{12}u_y + C_{13}u_z + C_{21}v_x + C_{22}v_y + C_{23}v_z \\ + C_{31}w_x + C_{32}w_y + C_{33}w_z = 0$$

$$(2.3) \quad D_{11}u_x + D_{12}u_y + D_{13}u_z + D_{21}v_x + D_{22}v_y + D_{23}v_z \\ + D_{31}w_x + D_{32}w_y + D_{33}w_z = 0$$

$$(2.4) \quad A_{11}u_x + A_{12}u_y + A_{13}u_z + A_{21}v_x + A_{22}v_y + A_{23}v_z \\ + A_{31}w_x + A_{32}w_y + A_{33}w_z = 0$$

or, in tensor notation, equations (2) may be written

$$B_{\beta\gamma}^k U_{\gamma}^{\beta} = 0 \quad (B^k = B, C, D, A; U^{\beta} = u, v, w; \gamma = x, y, z)$$

then under what conditions can we transform them into the Newtonian equations? For clearness and simplicity, we shall take up the problem in three parts: the first is to transform (2.1), (2.2) and (2.3) into rotation equations; the second is to take four linear combinations of (2) and transform three of the four combinations into rotation equations; and the third is to transform four equations, the four linear combinations in part second which already underwent transformations that brought three of them into rotation equations, into Newtonian equations.

Part First

Rotation Equations

2. Transformation of functions and variables. Let our formulas of transformation on the functions u' , v' , w' and the variables x' , y' , z' be

$$(3.1) \begin{cases} u' = a_{11} u + a_{12} v + a_{13} w \\ v' = a_{21} u + a_{22} v + a_{23} w \\ w' = a_{31} u + a_{32} v + a_{33} w \end{cases} \quad (3.2) \begin{cases} u = a'_{11} u' + a'_{12} v' + a'_{13} w' \\ v = a'_{21} u' + a'_{22} v' + a'_{23} w' \\ w = a'_{31} u' + a'_{32} v' + a'_{33} w' \end{cases}$$

$$(3.3) \begin{cases} x = b_{11} x' + b_{12} y' + b_{13} z' \\ y = b_{21} x' + b_{22} y' + b_{23} z' \\ z = b_{31} x' + b_{32} y' + b_{33} z' \end{cases} \quad (3.4) \begin{cases} x' = b'_{11} x + b'_{12} y + b'_{13} z \\ y' = b'_{21} x + b'_{22} y + b'_{23} z \\ z' = b'_{31} x + b'_{32} y + b'_{33} z \end{cases}$$

where (3.2) and (3.4) are inverse transformations. By the ordinary rule of partial differentiation under change of variables we can easily verify that equations (1.1), (1.2) and (1.3) will become, by means of (3.1) and (3.3),

$$(4.1) \quad (a_{11}b_{12} - a_{21}b_{11}) u_x + (a_{11}b_{22} - a_{21}b_{21}) u_y + (a_{11}b_{32} - a_{21}b_{31}) u_z \\ + (a_{12}b_{12} - a_{22}b_{11}) v_x + (a_{12}b_{22} - a_{22}b_{21}) v_y + (a_{12}b_{32} - a_{22}b_{31}) v_z \\ + (a_{13}b_{12} - a_{23}b_{11}) w_x + (a_{13}b_{22} - a_{23}b_{21}) w_y + (a_{13}b_{32} - a_{23}b_{31}) w_z = 0$$

$$(4.2) \quad (a_{11}b_{13} - a_{31}b_{11}) u_x + (a_{11}b_{23} - a_{31}b_{21}) u_y + (a_{11}b_{33} - a_{31}b_{31}) u_z \\ + (a_{12}b_{13} - a_{32}b_{11}) v_x + (a_{12}b_{23} - a_{32}b_{21}) v_y + (a_{12}b_{33} - a_{32}b_{31}) v_z \\ + (a_{13}b_{13} - a_{33}b_{11}) w_x + (a_{13}b_{23} - a_{33}b_{21}) w_y + (a_{13}b_{33} - a_{33}b_{31}) w_z = 0$$

$$(4.3) \quad (a_{21}b_{13} - a_{31}b_{12}) u_x + (a_{21}b_{23} - a_{31}b_{22}) u_y + (a_{21}b_{33} - a_{31}b_{32}) u_z \\ + (a_{22}b_{13} - a_{32}b_{12}) v_x + (a_{22}b_{23} - a_{32}b_{22}) v_y + (a_{22}b_{33} - a_{32}b_{32}) v_z \\ + (a_{23}b_{13} - a_{33}b_{12}) w_x + (a_{23}b_{23} - a_{33}b_{22}) w_y + (a_{23}b_{33} - a_{33}b_{32}) w_z = 0$$

Here we know that equations (4) can be brought to the rotation equations simply by inverse transformation, that is, by (3.2) and (3.4). Obviously, equations which are identical with (4) must be transformable into rotation equations. Therefore, we wish to identify coefficients of partial derivatives in (2.1), (2.2) and (2.3) with those of (4.1), (4.2) and (4.3) respectively. In this manner we obtain 27 equations involving 18 introduced constants a_{ij} and b_{ij} ($i, j=1, 2, 3$) which we can choose arbitrarily,

$$(5.1) \begin{cases} a_{11} b_{12} - a_{21} b_{11} = B_{11} \\ a_{11} b_{13} - a_{31} b_{11} = C_{11} \end{cases} \quad (5.2) \begin{cases} a_{11} b_{12} - a_{21} b_{21} = B_{12} \\ a_{11} b_{23} - a_{31} b_{21} = C_{12} \end{cases}$$

$$(5.3) \begin{cases} a_{11} b_{32} - a_{21} b_{31} = B_{13} \\ a_{11} b_{33} - a_{31} b_{31} = C_{13} \end{cases} \quad (5.4) \begin{cases} a_{12} b_{12} - a_{22} b_{11} = B_{21} \\ a_{12} b_{13} - a_{32} b_{11} = C_{21} \end{cases}$$

$$(5.5) \begin{cases} a_{12} b_{22} - a_{22} b_{21} = B_{22} \\ a_{12} b_{23} - a_{32} b_{21} = C_{22} \end{cases} \quad (5.6) \begin{cases} a_{12} b_{32} - a_{22} b_{31} = B_{23} \\ a_{12} b_{33} - a_{32} b_{31} = C_{23} \end{cases}$$

$$(5.7) \begin{cases} a_{13} b_{12} - a_{23} b_{11} = B_{31} \\ a_{13} b_{13} - a_{33} b_{11} = C_{31} \end{cases}$$

$$(5.8) \begin{cases} a_{13} b_{22} - a_{23} b_{21} = B_{32} \\ a_{13} b_{23} - a_{33} b_{21} = C_{32} \end{cases}$$

$$(5.9) \begin{cases} a_{13} b_{32} - a_{23} b_{31} = B_{33} \\ a_{13} b_{33} - a_{33} b_{31} = C_{33} \end{cases}$$

$$(6.1) a_{21} b_{13} - a_{31} b_{12} = D_{11}$$

$$(6.2) a_{21} b_{23} - a_{31} b_{22} = D_{12}$$

$$(6.3) a_{21} b_{33} - a_{31} b_{32} = D_{13}$$

$$(6.4) a_{22} b_{13} - a_{32} b_{12} = D_{21}$$

$$(6.5) a_{22} b_{23} - a_{32} b_{22} = D_{22}$$

$$(6.6) a_{22} b_{33} - a_{32} b_{32} = D_{23}$$

$$(6.7) a_{23} b_{13} - a_{33} b_{12} = D_{31}$$

$$(6.8) a_{23} b_{23} - a_{33} b_{22} = D_{32}$$

$$(6.9) a_{23} b_{33} - a_{33} b_{32} = D_{33}$$

3. Solving for a_{ij} and b_{ij} ($i, j=1, 2, 3$) in 27 equations. If we are able to find a_{ij} and b_{ij} that will satisfy the above 27 equations (5) and (6), then, it means that we can find linear transformations (3.1) and (3.3) that will bring (2.1), (2.2) and (2.3) into rotation equations; hence, the first part of our problem will be solved. Therefore, we shall proceed to solve the 27 equations immediately.

From (5.1) we can solve for a_{11} and b_{11} by Gramer's rule,

$$a_{11} = \frac{\begin{vmatrix} B_{11} & -a_{21} \\ C_{11} & -a_{31} \end{vmatrix}}{\begin{vmatrix} b_{12} & -a_{21} \\ b_{13} & -a_{31} \end{vmatrix}}, \quad b_{11} = \frac{\begin{vmatrix} b_{12} & B_{11} \\ b_{13} & C_{11} \end{vmatrix}}{\begin{vmatrix} b_{12} & -a_{21} \\ b_{13} & -a_{31} \end{vmatrix}}$$

Since their denominators happen to be the left-hand member of (6.1), therefore, we can substitute D_{11} for the expression $b_{13}a_{21} - b_{12}a_{31}$ and thereby, we get one linear equation in a_{11} and another in b_{11} .

In similar manner, (5.2), (5.3), ..., and (5.9) can be reduced, by means of (6), to linear equations either in a_{ij} alone or in b_{ij} alone. Of course, a_{ij} and b_{ij} found this way will have to satisfy equations (6) in order to satisfy all 27 equations. The complete results of reduction, as we have just illustrated, in (5) may be put below:

$$(7.1) \quad \begin{cases} a_{11} D_{11} - a_{21} C_{11} + a_{31} B_{11} = 0 \\ a_{11} D_{12} - a_{21} C_{12} + a_{31} B_{12} = 0 \\ a_{11} D_{13} - a_{21} C_{13} + a_{31} B_{13} = 0 \end{cases}$$

$$(7.2) \quad \begin{cases} a_{12} D_{21} - a_{22} C_{21} + a_{32} B_{21} = 0 \\ a_{12} D_{22} - a_{22} C_{22} + a_{32} B_{22} = 0 \\ a_{12} D_{23} - a_{22} C_{23} + a_{32} B_{23} = 0 \end{cases}$$

$$(7.3) \quad \begin{cases} a_{13} D_{31} - a_{23} C_{31} + a_{33} B_{31} = 0 \\ a_{13} D_{32} - a_{23} C_{32} + a_{33} B_{32} = 0 \\ a_{13} D_{33} - a_{23} C_{33} + a_{33} B_{33} = 0 \end{cases}$$

$$(7.4) \quad \begin{cases} b_{11} D_{11} - b_{12} C_{11} + b_{13} B_{11} = 0 \\ b_{11} D_{21} - b_{12} C_{21} + b_{13} B_{21} = 0 \\ b_{11} D_{31} - b_{12} C_{31} + b_{13} B_{31} = 0 \end{cases}$$

$$(7.5) \quad \begin{cases} b_{21} D_{12} - b_{22} C_{12} + b_{23} B_{12} = 0 \\ b_{21} D_{22} - b_{22} C_{22} + b_{23} B_{22} = 0 \\ b_{21} D_{32} - b_{22} C_{32} + b_{23} B_{32} = 0 \end{cases}$$

$$(7.6) \quad \begin{cases} b_{31} D_{13} - b_{32} C_{13} + b_{33} B_{13} = 0 \\ b_{31} D_{23} - b_{32} C_{23} + b_{33} B_{23} = 0 \\ b_{31} D_{33} - b_{32} C_{33} + b_{33} B_{33} = 0 \end{cases}$$

4. Ten conditions necessary and sufficient for equations (5) and (6) to have a solution. Thus each of the six systems of equations in (7), being homogeneous, can be solved (zero solution excluded) when, and only when,

$$(8.1) \quad \begin{vmatrix} D_{11} & -C_{11} & B_{11} \\ D_{12} & -C_{12} & B_{12} \\ D_{13} & -C_{13} & B_{13} \end{vmatrix} = 0$$

$$(8.2) \quad \begin{vmatrix} D_{21} & -C_{21} & B_{21} \\ D_{22} & -C_{22} & B_{22} \\ D_{23} & -C_{23} & B_{23} \end{vmatrix} = 0$$

$$(8.3) \quad \begin{vmatrix} D_{31} & -C_{31} & B_{31} \\ D_{32} & -C_{32} & B_{32} \\ D_{33} & -C_{33} & B_{33} \end{vmatrix} = 0$$

$$(8.4) \quad \begin{vmatrix} D_{11} & -C_{11} & B_{11} \\ D_{21} & -C_{21} & B_{21} \\ D_{31} & -C_{31} & B_{31} \end{vmatrix} = 0$$

$$(8.5) \quad \begin{vmatrix} D_{12} & -C_{12} & B_{12} \\ D_{22} & -C_{22} & B_{22} \\ D_{32} & -C_{32} & B_{32} \end{vmatrix} = 0$$

$$(8.6) \quad \begin{vmatrix} D_{13} & -C_{13} & B_{13} \\ D_{23} & -C_{23} & B_{23} \\ D_{33} & -C_{33} & B_{33} \end{vmatrix} = 0$$

From theory of determinants we learn that the cofactors of elements of any row in (8.1), (8.2), ..., and (8.6) will satisfy (7.1), (7.2), ..., and (7.6) respectively. But the equa-

tions in each of the six systems (7) are homogeneous, therefore, every solution of one system multiplied by a constant, different from zero, is still a solution. For the six systems we can introduce six arbitrary constants, say, r_i and s_i ($i=1,2,3$). If now, we take cofactors of elements of first rows in determinants (8) and multiply them by arbitrary constants we get a_{ij} and b_{ij} as follows:

$$\begin{aligned}
 (9) \quad & a_{11}=r_1 \begin{vmatrix} C_{12} & B_{12} \\ C_{13} & B_{13} \end{vmatrix} & a_{21}=r_1 \begin{vmatrix} D_{12} & B_{12} \\ D_{13} & B_{13} \end{vmatrix} \\
 & a_{31}=r_1 \begin{vmatrix} D_{12} & C_{12} \\ D_{13} & C_{13} \end{vmatrix} & \\
 & a_{12}=r_2 \begin{vmatrix} C_{22} & B_{22} \\ C_{23} & B_{23} \end{vmatrix} & a_{22}=r_2 \begin{vmatrix} D_{22} & B_{22} \\ D_{23} & B_{23} \end{vmatrix} \\
 & a_{32}=r_2 \begin{vmatrix} D_{22} & C_{22} \\ D_{23} & C_{23} \end{vmatrix} & \\
 & a_{13}=r_3 \begin{vmatrix} D_{32} & B_{32} \\ C_{33} & B_{33} \end{vmatrix} & a_{23}=r_3 \begin{vmatrix} D_{32} & B_{32} \\ D_{33} & B_{33} \end{vmatrix} \\
 & a_{33}=r_3 \begin{vmatrix} D_{32} & C_{32} \\ D_{33} & C_{33} \end{vmatrix} & \\
 (10) \quad & b_{11}=s_1 \begin{vmatrix} C_{21} & B_{21} \\ C_{31} & B_{31} \end{vmatrix} & b_{12}=s_1 \begin{vmatrix} D_{21} & B_{21} \\ D_{31} & B_{31} \end{vmatrix} \\
 & b_{13}=s_1 \begin{vmatrix} D_{21} & C_{21} \\ D_{31} & C_{31} \end{vmatrix} & \\
 & b_{21}=s_2 \begin{vmatrix} C_{22} & B_{22} \\ C_{32} & B_{32} \end{vmatrix} & b_{22}=s_2 \begin{vmatrix} D_{22} & B_{22} \\ D_{32} & B_{32} \end{vmatrix} \\
 & b_{23}=s_2 \begin{vmatrix} D_{22} & C_{22} \\ D_{32} & C_{32} \end{vmatrix} & \\
 & b_{31}=s_3 \begin{vmatrix} C_{23} & B_{23} \\ C_{33} & B_{33} \end{vmatrix} & b_{32}=s_3 \begin{vmatrix} D_{23} & B_{23} \\ D_{33} & B_{33} \end{vmatrix} \\
 & b_{33}=s_3 \begin{vmatrix} D_{23} & C_{23} \\ D_{33} & C_{33} \end{vmatrix} &
 \end{aligned}$$

It now only remains to see whether a_{ij} and b_{ij} thus found, do satisfy the nine equations in (6) or not, which can be proved by substitution. On eliminating r_i and s_i , being arbitrary, we find four conditions on the coefficients of (1.1), (1.2) and (1.3),

$$(11) \quad \begin{cases} \frac{D_{12} M_{11}}{D_{11} M_{12}} = \frac{D_{22} M_{21}}{D_{21} M_{22}} = \frac{D_{32} M_{31}}{D_{31} M_{32}} \\ \frac{D_{23} M_{22}}{D_{22} M_{23}} = \frac{D_{13} M_{12}}{D_{12} M_{13}} = \frac{D_{33} M_{32}}{D_{32} M_{33}} \end{cases}$$

where

$$\begin{aligned} M_{11} &= \begin{vmatrix} D_{12} & B_{12} \\ D_{13} & B_{13} \end{vmatrix} \cdot \begin{vmatrix} D_{21} & C_{21} \\ D_{31} & C_{31} \end{vmatrix} - \begin{vmatrix} D_{12} & C_{12} \\ D_{13} & C_{13} \end{vmatrix} \cdot \begin{vmatrix} D_{21} & B_{21} \\ D_{31} & B_{31} \end{vmatrix} \\ M_{12} &= \begin{vmatrix} D_{12} & B_{12} \\ D_{13} & B_{13} \end{vmatrix} \cdot \begin{vmatrix} D_{22} & C_{22} \\ D_{32} & C_{32} \end{vmatrix} - \begin{vmatrix} D_{12} & C_{12} \\ D_{13} & C_{13} \end{vmatrix} \cdot \begin{vmatrix} D_{22} & B_{22} \\ D_{32} & B_{32} \end{vmatrix} \\ M_{13} &= \begin{vmatrix} D_{12} & B_{12} \\ D_{13} & B_{13} \end{vmatrix} \cdot \begin{vmatrix} D_{23} & C_{23} \\ D_{33} & C_{33} \end{vmatrix} - \begin{vmatrix} D_{12} & C_{12} \\ D_{13} & C_{13} \end{vmatrix} \cdot \begin{vmatrix} D_{23} & B_{23} \\ D_{33} & B_{33} \end{vmatrix} \\ M_{21} &= \begin{vmatrix} D_{22} & B_{22} \\ D_{23} & B_{23} \end{vmatrix} \cdot \begin{vmatrix} D_{21} & C_{21} \\ D_{31} & C_{31} \end{vmatrix} - \begin{vmatrix} D_{22} & C_{22} \\ D_{23} & C_{23} \end{vmatrix} \cdot \begin{vmatrix} D_{21} & B_{21} \\ D_{31} & B_{31} \end{vmatrix} \\ M_{22} &= \begin{vmatrix} D_{22} & B_{22} \\ D_{23} & B_{23} \end{vmatrix} \cdot \begin{vmatrix} D_{22} & C_{22} \\ D_{32} & C_{32} \end{vmatrix} - \begin{vmatrix} D_{22} & C_{22} \\ D_{23} & C_{23} \end{vmatrix} \cdot \begin{vmatrix} D_{22} & B_{22} \\ D_{32} & B_{32} \end{vmatrix} \\ M_{23} &= \begin{vmatrix} D_{22} & B_{22} \\ D_{23} & B_{23} \end{vmatrix} \cdot \begin{vmatrix} D_{23} & C_{23} \\ D_{33} & C_{33} \end{vmatrix} - \begin{vmatrix} D_{22} & C_{22} \\ D_{23} & C_{23} \end{vmatrix} \cdot \begin{vmatrix} D_{23} & B_{23} \\ D_{33} & B_{33} \end{vmatrix} \\ M_{31} &= \begin{vmatrix} D_{32} & B_{32} \\ D_{33} & B_{33} \end{vmatrix} \cdot \begin{vmatrix} D_{21} & C_{21} \\ D_{31} & C_{31} \end{vmatrix} - \begin{vmatrix} D_{32} & C_{32} \\ D_{33} & C_{33} \end{vmatrix} \cdot \begin{vmatrix} D_{21} & B_{21} \\ D_{31} & B_{31} \end{vmatrix} \\ M_{32} &= \begin{vmatrix} D_{32} & B_{32} \\ D_{33} & B_{33} \end{vmatrix} \cdot \begin{vmatrix} D_{22} & C_{22} \\ D_{32} & C_{32} \end{vmatrix} - \begin{vmatrix} D_{32} & C_{32} \\ D_{33} & C_{33} \end{vmatrix} \cdot \begin{vmatrix} D_{22} & B_{22} \\ D_{32} & B_{32} \end{vmatrix} \\ M_{33} &= \begin{vmatrix} D_{32} & B_{32} \\ D_{33} & B_{33} \end{vmatrix} \cdot \begin{vmatrix} D_{23} & C_{23} \\ D_{33} & C_{33} \end{vmatrix} - \begin{vmatrix} D_{32} & C_{32} \\ D_{33} & C_{33} \end{vmatrix} \cdot \begin{vmatrix} D_{23} & B_{23} \\ D_{33} & B_{33} \end{vmatrix} \end{aligned}$$

It is our purpose to simplify M_{ij} if possible. Multiplying out the expression for M_{11} we get

$$\begin{aligned} M_{11} &= D_{12} B_{13} (D_{21} C_{31} - C_{21} D_{31}) - C_{13} (D_{21} B_{31} - D_{31} B_{21}) \\ &\quad - D_{13} B_{12} (D_{21} C_{31} - C_{21} D_{31}) - C_{12} (D_{21} B_{31} - D_{31} B_{21}) \end{aligned}$$

To the last expression we may add and subtract the quantity

$$D_{12} D_{13} \begin{vmatrix} B_{31} & B_{21} \\ C_{31} & C_{21} \end{vmatrix}$$

to complete third order determinants so that

$$M_{11}=D_{12} \begin{vmatrix} B_{13} & B_{31} & B_{21} \\ C_{13} & C_{31} & C_{21} \\ D_{13} & D_{31} & D_{21} \end{vmatrix} - D_{13} \begin{vmatrix} B_{12} & B_{31} & B_{21} \\ C_{12} & C_{31} & C_{21} \\ D_{12} & D_{31} & D_{21} \end{vmatrix}$$

In similar manner we can simplify the remaining M_{ij} which may be put as follows

$$\begin{array}{ll} M_{12}=D_{12} \begin{vmatrix} B_{22} & B_{13} & B_{32} \\ C_{22} & C_{13} & C_{32} \\ D_{22} & D_{13} & D_{32} \end{vmatrix} & M_{13}=D_{13} \begin{vmatrix} B_{12} & B_{23} & B_{33} \\ C_{12} & C_{23} & C_{33} \\ D_{12} & D_{23} & D_{33} \end{vmatrix} \\ M_{21}=D_{21} \begin{vmatrix} B_{22} & B_{23} & B_{31} \\ C_{22} & C_{23} & C_{31} \\ D_{22} & D_{23} & D_{31} \end{vmatrix} & M_{22}=D_{22} \begin{vmatrix} B_{22} & B_{23} & B_{32} \\ C_{22} & C_{23} & C_{32} \\ D_{22} & D_{23} & D_{32} \end{vmatrix} \\ M_{23}=D_{23} \begin{vmatrix} B_{22} & B_{23} & B_{33} \\ C_{22} & C_{23} & C_{33} \\ D_{22} & D_{23} & D_{33} \end{vmatrix} & M_{31}=D_{31} \begin{vmatrix} B_{33} & B_{32} & B_{21} \\ C_{33} & C_{32} & C_{21} \\ D_{33} & D_{32} & D_{21} \end{vmatrix} \\ M_{32}=D_{32} \begin{vmatrix} B_{22} & B_{33} & B_{32} \\ C_{22} & C_{33} & C_{32} \\ D_{22} & D_{33} & D_{32} \end{vmatrix} & M_{33}=D_{33} \begin{vmatrix} B_{32} & B_{23} & B_{33} \\ C_{32} & C_{23} & C_{33} \\ D_{32} & D_{23} & D_{33} \end{vmatrix} \end{array}$$

with the above simplifications on M_{ij} conditions (11) can be written as

$$(12) \quad \frac{D_{12}}{D_{11}} \begin{vmatrix} B_{13} & B_{31} & B_{21} \\ C_{13} & C_{31} & C_{21} \\ D_{13} & D_{31} & D_{21} \end{vmatrix} - \frac{D_{13}}{D_{11}} \begin{vmatrix} B_{12} & B_{31} & B_{21} \\ C_{12} & C_{31} & C_{21} \\ D_{12} & D_{31} & D_{21} \end{vmatrix}$$

$$\begin{vmatrix} B_{13} & B_{32} & B_{22} \\ C_{13} & C_{32} & C_{22} \\ D_{13} & D_{32} & D_{22} \end{vmatrix}$$

$$= \frac{\begin{vmatrix} B_{22} & B_{23} & B_{31} \\ C_{22} & C_{23} & C_{31} \\ D_{22} & D_{23} & D_{31} \end{vmatrix}}{\begin{vmatrix} B_{22} & B_{23} & B_{32} \\ C_{22} & C_{23} & C_{32} \\ D_{22} & D_{23} & D_{32} \end{vmatrix}} = \frac{\begin{vmatrix} B_{33} & B_{32} & B_{21} \\ C_{33} & C_{32} & C_{21} \\ D_{33} & D_{32} & D_{31} \end{vmatrix}}{\begin{vmatrix} B_{33} & B_{32} & B_{22} \\ C_{33} & C_{32} & C_{22} \\ D_{33} & D_{32} & D_{22} \end{vmatrix}}$$

$$\begin{vmatrix} B_{22} & B_{23} & B_{32} \\ C_{22} & C_{23} & C_{32} \\ D_{22} & D_{23} & D_{32} \end{vmatrix} = \begin{vmatrix} B_{13} & B_{32} & B_{22} \\ C_{13} & C_{32} & C_{22} \\ D_{13} & D_{32} & D_{22} \end{vmatrix} = \begin{vmatrix} B_{33} & B_{32} & B_{22} \\ C_{33} & C_{32} & C_{22} \\ D_{33} & D_{32} & D_{22} \end{vmatrix} \\
 \begin{vmatrix} B_{22} & B_{23} & B_{33} \\ C_{32} & C_{23} & C_{33} \\ D_{22} & D_{23} & D_{33} \end{vmatrix} = \begin{vmatrix} B_{12} & B_{23} & B_{33} \\ C_{12} & C_{23} & C_{33} \\ D_{12} & D_{23} & D_{33} \end{vmatrix} = \begin{vmatrix} B_{32} & B_{23} & B_{33} \\ C_{32} & C_{23} & C_{33} \\ D_{32} & D_{23} & D_{33} \end{vmatrix}$$

Thus, the four conditions in (12) and the six conditions in (8) constitute ten conditions necessary and sufficient for equations (5) and (6) to be solvable in a_{ij} and b_{ij} , that is, for equations (2.1), (2.2) and (2.3) to be identical with equations (4) which are known to be transformable into rotations. In short, our transformation formulas (3) which consist of a_{ij} and b_{ij} thus found can not fail to carry (2.1), (2.2) and (2.3) over into three rotation equations. Hence,

Theorem: *The ten conditions (8) and (12) are necessary and sufficient for the transformability of three given equations into rotation equations.*

5. Transformation of three equations. So far we have transformed the functions and variables and ten conditions have been found necessary and sufficient for a transformability of three equations into rotation equations. Now, we shall try to transform the given equations (2.1), (2.2) and (2.3) through the matrix $\|c_{ij}\|$ ($i, j=1, 2, 3$) and get three linear combinations,

$$(13) \quad \begin{cases} c_{11}B + c_{12}C + c_{13}D = 0 \\ c_{21}B + c_{22}C + c_{23}D = 0 \\ c_{31}B + c_{32}C + c_{33}D = 0 \end{cases}$$

where B , C and D stand for the left-hand members of (2.1), (2.2) and (2.3) respectively.

If equations (13) are transformable into rotation equations then their coefficients must satisfy the conditions in (8) and (12). For instance, (8.1) will become (14):

$$\begin{vmatrix} (c_{11}B_{11} + c_{12}C_{11} + c_{13}D_{11}) & (c_{21}B_{11} + c_{22}C_{11} + c_{23}D_{11}) & (c_{31}B_{11} + c_{32}C_{11} + c_{33}D_{11}) \\ (c_{11}B_{12} + c_{12}C_{12} + c_{13}D_{12}) & (c_{21}B_{12} + c_{22}C_{12} + c_{23}D_{12}) & (c_{31}B_{12} + c_{32}C_{12} + c_{33}D_{12}) \\ (c_{11}B_{13} + c_{12}C_{13} + c_{13}D_{13}) & (c_{21}B_{13} + c_{22}C_{13} + c_{23}D_{13}) & (c_{31}B_{13} + c_{32}C_{13} + c_{33}D_{13}) \end{vmatrix}$$

$$= \begin{vmatrix} B_{11} & C_{11} & D_{11} \\ B_{12} & C_{12} & D_{12} \\ B_{13} & C_{13} & D_{13} \end{vmatrix} \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 0$$

through multiplication of determinants. But the determinant c_{ij} ($i, j=1, 2, 3$) is not zero by hypothesis in our transformation, therefore,

$$\begin{vmatrix} B_{11} & C_{11} & D_{11} \\ B_{12} & C_{12} & D_{12} \\ B_{13} & C_{13} & D_{13} \end{vmatrix} = 0$$

which is (8.1) again. In like manner, the remaining nine conditions in (8) and (12) will not be altered by the new coefficients of transformed equations (13).

So, the linear combinations in (13) are under exactly the same necessary and sufficient condition for transformability into rotation equations as equations (1.1), (1.2) and (1.3) are. In other words, *a transformation of the three given equations has failed to reduce our conditions or to give anything new.* In the next section we will transform four given equations instead of three.

Part Second

Linear Combinations of Four Equations

6. Four linear combinations. From the very start we were given four equations; so, whenever we wish to transform them the most general transformation consists of linear combinations of all four equations. If we denote our transformation matrix by

$$(15) \quad \begin{vmatrix} d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \\ d_{11} & d_{12} & d_{13} & d_{14} \end{vmatrix}$$

and our transformed equations by

$$\begin{aligned}
 (16) \quad & B' = d_{21}A + d_{22}B + d_{23}C + d_{24}D = 0 \\
 & C' = d_{31}A + d_{32}B + d_{33}C + d_{34}D = 0 \\
 & D' = d_{41}A + d_{42}B + d_{43}C + d_{44}D = 0 \\
 & A' = d_{11}A + d_{12}B + d_{13}C + d_{14}D = 0
 \end{aligned}$$

where B, C, D and A stand for the left-hand members of (2), then the new coefficients in B', C', D' and A' may be denoted by $B'_{ij}, C'_{ij}, D'_{ij}$ and A'_{ij} ($i, j=1, 2, 3$) respectively.

7. Elimination of three conditions. If we wish to transform equations (16.1), (16.2) and (16.3) into rotation equations, then their coefficients must satisfy the necessary and sufficient condition for transformability as proved in Section 4. By virtue of our introduced constants we hope to eliminate some of those ten conditions (8), (12) the first of which may be written

$$\begin{aligned}
 & \begin{vmatrix} B'_{11} & C'_{11} & D'_{11} \\ B'_{12} & C'_{12} & D'_{12} \\ B'_{13} & C'_{13} & D'_{13} \end{vmatrix} = \begin{vmatrix} B_{11} & C_{11} & D_{11} \\ B_{12} & C_{12} & D_{12} \\ A_{13} & C_{13} & D_{13} \end{vmatrix} \cdot \begin{vmatrix} d_{22} & d_{32} & d_{42} \\ d_{23} & d_{33} & d_{43} \\ d_{24} & d_{34} & d_{44} \end{vmatrix} \\
 & - \begin{vmatrix} A_{11} & C_{11} & D_{11} \\ A_{12} & C_{12} & D_{12} \\ B_{13} & C_{13} & D_{13} \end{vmatrix} \cdot \begin{vmatrix} d_{21} & d_{31} & d_{41} \\ d_{24} & d_{34} & d_{44} \\ d_{23} & d_{33} & d_{43} \end{vmatrix} + \begin{vmatrix} A_{11} & B_{11} & C_{11} \\ A_{12} & B_{12} & C_{12} \\ A_{13} & B_{13} & C_{13} \end{vmatrix} \cdot \begin{vmatrix} d_{21} & d_{31} & d_{41} \\ d_{22} & d_{32} & d_{42} \\ d_{24} & d_{34} & d_{44} \end{vmatrix} \\
 & - \begin{vmatrix} A_{11} & B_{11} & D_{11} \\ A_{12} & B_{12} & D_{12} \\ A_{13} & B_{13} & D_{13} \end{vmatrix} \cdot \begin{vmatrix} d_{21} & d_{31} & d_{41} \\ d_{23} & d_{33} & d_{43} \\ d_{22} & d_{32} & d_{42} \end{vmatrix} = 0
 \end{aligned}$$

$$(17.1) \quad = F_{11}G_1 + F_{12}G_2 + F_{13}G_3 + F_{14}G_4 = 0$$

and five more of similar type from (8)

$$(17.2) \quad F_{21}G_1 + F_{22}G_2 + F_{23}G_3 + F_{24}G_4 = 0$$

$$(17.3) \quad F_{31}G_1 + F_{32}G_2 + F_{33}G_3 + F_{34}G_4 = 0$$

$$(17.4) \quad F_{41}G_1 + F_{42}G_2 + F_{43}G_3 + F_{44}G_4 = 0$$

$$(17.5) \quad F_{51}G_1 + F_{52}G_2 + F_{53}G_3 + F_{54}G_4 = 0$$

$$(17.6) \quad F_{61}G_1 + F_{62}G_2 + F_{63}G_3 + F_{64}G_4 = 0$$

where G_k ($k=1, 2, 3, 4$) is a determinant obtained from the matrix

$$\begin{vmatrix} d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{vmatrix}$$

by omitting the k th column of elements; and F_{hk} [$h=1,2,3,4$, 5,6 in the order of the conditions (6.1), ..., (6.6)], are functions of some three of A_{ij} , B_{ij} , C_{ij} and D_{ij} .

In order to solve for G_k from the first four equations of system (17) it is necessary and sufficient

$$(18.1) \quad \begin{vmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{vmatrix} = 0$$

As soon as G_k are known then we can find a solution in d_{mn} ($m,n=2,3,4$) satisfying G_1 and solve for the remaining d_{21} , d_{31} and d_{41} linearly from G_2 , G_3 and G_4 . With G_k obtained, say, as cofactors of elements F_{1k} from the determinant of (18.1) we can substitute them in (17.5) and (17.6) which become

$$(18.2) \quad \begin{vmatrix} F_{51} & F_{52} & F_{53} & F_{54} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{vmatrix} = 0$$

$$(18.3) \quad \begin{vmatrix} F_{61} & F_{62} & F_{63} & F_{64} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{vmatrix} = 0$$

8. Seven conditions necessary and sufficient for transformability into rotation equations. The three conditions (18) are therefore, necessary and sufficient for system (17) to admit a solution in G_k , which means that the six conditions in (8) are fulfilled. The remaining four conditions in (12) must be satisfied in order to satisfy all ten conditions which are necessary and sufficient for a successful transformation into rotation equations. Hence, we substitute A'_{ij} , B'_{ij} , C'_{ij} and D'_{ij} in place of A_{ij} , B_{ij} , C_{ij} and D_{ij} in (12) and get

$$\begin{aligned}
 & \begin{array}{c} 1 \\ D'_{11} \end{array} \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ H_1 & H_2 & H_3 & H_4 \\ A_{31} & B_{31} & C_{31} & D_{31} \\ A_{21} & B_{21} & C_{21} & D_{21} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{22} & B_{22} & C_{22} & D_{22} \\ A_{23} & B_{23} & C_{23} & D_{23} \\ A_{31} & B_{31} & C_{31} & D_{31} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{33} & B_{33} & C_{33} & D_{33} \\ A_{32} & B_{32} & C_{32} & D_{32} \\ A_{21} & B_{21} & C_{21} & D_{21} \end{vmatrix} \\
 (19) \quad & \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{13} & B_{13} & C_{13} & D_{13} \\ A_{32} & B_{32} & C_{32} & D_{32} \\ A_{22} & B_{22} & C_{22} & D_{22} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{22} & B_{22} & C_{22} & D_{22} \\ A_{23} & B_{23} & C_{23} & D_{23} \\ A_{32} & B_{32} & C_{32} & D_{32} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{33} & B_{33} & C_{33} & D_{33} \\ A_{32} & B_{32} & C_{32} & D_{32} \\ A_{22} & B_{22} & C_{22} & D_{22} \end{vmatrix} \\
 & \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{22} & B_{22} & C_{22} & D_{22} \\ A_{23} & B_{23} & C_{23} & D_{23} \\ A_{32} & B_{32} & C_{32} & D_{32} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{13} & B_{13} & C_{13} & D_{13} \\ A_{32} & B_{32} & C_{32} & D_{32} \\ A_{22} & B_{22} & C_{22} & D_{22} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{33} & B_{33} & C_{33} & D_{33} \\ A_{32} & B_{32} & C_{32} & D_{32} \\ A_{22} & B_{22} & C_{22} & D_{22} \end{vmatrix} \\
 & \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{22} & B_{22} & C_{22} & D_{22} \\ A_{23} & B_{23} & C_{23} & D_{23} \\ A_{33} & B_{33} & C_{33} & D_{33} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{12} & B_{12} & C_{12} & D_{12} \\ A_{23} & B_{23} & C_{23} & D_{23} \\ A_{33} & B_{33} & C_{33} & D_{33} \end{vmatrix} = \begin{vmatrix} G_1 & G_2 & G_3 & G_4 \\ A_{32} & B_{32} & C_{32} & D_{32} \\ A_{23} & B_{23} & C_{23} & D_{23} \\ A_{33} & B_{33} & C_{33} & D_{33} \end{vmatrix}
 \end{aligned}$$

where

$$\begin{aligned}
 H_1 &= A_{13}D'_{12} - A_{12}D'_{13} \\
 H_2 &= B_{13}D'_{12} - B_{12}D'_{13} \\
 H_3 &= C_{13}D'_{12} - C_{12}D'_{13} \\
 H_4 &= D_{13}D'_{12} - D_{12}D'_{13} \\
 D'_{11} &= d_{41}A_{11} + d_{42}B_{11} + d_{43}C_{11} + d_{44}D_{11} \\
 D'_{12} &= d_{41}A_{12} + d_{42}B_{12} + d_{43}C_{12} + d_{44}D_{12} \\
 D'_{13} &= d_{41}A_{13} + d_{42}B_{13} + d_{43}C_{13} + d_{44}D_{13}
 \end{aligned}$$

Thus we may state the following

Theorem; *The four conditions in (19) and three in (18) make seven conditions, necessary and sufficient, on the coefficients of (2) such that three of four linear combinations of which can be transformed into rotation equations.*

Part Third

Divergence Equation

9. Explanation of the problem. Assume that the coefficients of four linear homogeneous differential equations do

satisfy the necessary and sufficient condition for transformability into rotation equations then we may transform them to the following

$$(20) \quad \begin{cases} u'_{y'} - v'_{x'} = 0 \\ u'_{z'} - w'_{x'} = 0 \\ v'_{z'} - w'_{y'} = 0 \end{cases}$$

and,

$$(21) \quad E_{11}u'_{x'} + E_{12}u'_{y'} + E_{13}u'_{z'} + E_{21}v'_{x'} + E_{22}v'_{y'} \\ + E_{23}v'_{z'} + E_{31}w'_{x'} + E_{32}w'_{y'} + E_{33}w'_{z'} = 0$$

where $E_{ij} = a'_{\alpha} \cdot A'_{\alpha\beta} \cdot b'_{j\beta}$ ($i, j = 1, 2, 3$; $\alpha, \beta = \Sigma 1, 2, 3$)
 $A'_{ij} = d_{11}A_{ij} + d_{12}B_{ij} + d_{13}C_{ij} + d_{14}D_{ij}$

Equation (21) is one of the original four linear combinations of (2), or, more precisely, it is (16.4) after having undergone transformation which successfully carried over (16.1), (16.2) and (16.3) into (20) although itself played no active part in that transformation into rotation equations. However, the constants d_{11} , d_{12} , d_{13} and d_{14} in (21) remain perfectly arbitrary so that we may choose their values whenever we want.

The present problem, then, is to find conditions under which we can transform equations (20) and (21) into Newtonian equations. In order to do that we have to apply again some or all of our three methods of transformation: transformation of variables, transformation of functions and transformation of equations.

10. Application of the theorem in Section 8 to the present problem. We learned in Section 8 that four given equations must satisfy certain conditions before they can be transformed into rotation equations which are a part of the Newtonian equations. Therefore, we must be able to transform equations (20) and (21) into rotation equations before we can transform them into Newtonian equations. To see if (20) and (21) satisfy the seven conditions in (18) and (19) which are

necessary and sufficient for their transformability into rotation equations we have to substitute the following values in (18) and (19)

$$(22) \quad \begin{cases} B_{12}=1 & B_{21}=-1 \\ C_{13}=1 & C_{31}=-1 \\ D_{23}=1 & D_{32}=-1 \\ \text{Remaining } B_{ij}, C_{ij} \text{ and } D_{ij} \text{ are all zero} \\ A_{ij}=E_{ij} \end{cases}$$

With the above known coefficients we can calculate F_{hk} ($h=1, 2, 3, 4, 5, 6$; $k=1, 2, 3, 4$) from their definitions (see Section 7)

$$(23) \quad \begin{cases} F_{11}=0 & F_{12}=0 & F_{13}=0 & F_{14}=E_{11} \\ F_{21}=0 & F_{22}=0 & F_{23}=E_{22} & F_{24}=0 \\ F_{31}=0 & F_{32}=E_{33} & F_{33}=0 & F_{34}=0 \\ F_{41}=0 & F_{42}=0 & F_{43}=0 & F_{44}=E_{11} \\ F_{51}=0 & F_{52}=0 & F_{53}=E_{22} & F_{54}=0 \\ F_{61}=0 & F_{62}=E_{33} & F_{63}=0 & F_{64}=0 \end{cases}$$

Substitute F_{hk} in condition (18.1) which becomes

$$(18.1') \quad \begin{vmatrix} 0 & 0 & 0 & E_{11} \\ 0 & 0 & E_{22} & 0 \\ 0 & E_{33} & 0 & 0 \\ 0 & 0 & 0 & E_{11} \end{vmatrix} = 0$$

whence we see that (18.1) is satisfied. From (18.1') we can evaluate G_k as cofactors of the elements of the top row:

$$(24) \quad G_1 = -E_{11} E_{22} E_{33}, \quad G_2 = G_3 = G_4 = 0$$

It can be seen by inspection that conditions (18.2), (18.3) and (19) will also be satisfied by (23) and (24).

From the values of G_k as given by (24) we want to solve for d_{2k} , d_{3k} and d_{4k} which we may suppose to be coefficients of the following three linear equations:

$$\begin{cases} d_{22}x_1 + d_{23}y_1 + d_{24}z_1 = d_{21} \\ d_{32}x_1 + d_{33}y_1 + d_{34}z_1 = d_{31} \\ d_{42}x_1 + d_{43}y_1 + d_{44}z_1 = d_{41} \end{cases}$$

By virtue of (24) and the method of Cramer's rule we find that the only solution of the above system of equations is

$$x_1=y_1=z_1=0$$

provided $G_1 \neq 0$. Therefore,

$$d_{21}=d_{31}=d_{41}=0$$

But d_{21} , d_{31} and d_{41} are coefficients of the left-hand expression of (21) in our three linear combinations in question (those three to be transformed into rotation equations), which are, therefore, free from representation of (21). Hence, three linear combinations of equations (20) are the only ones transformable into rotation equations, and no other linear combinations from the four equations in (20) and (21).

Now, if we can transform equations (20) into rotation equations and (21) into a divergence equation of the form (1.4) then our general problem raised in Section 1 will be completely solved, that is, conditions will be found under which the given equations (2) can be transformed into Newtonian equations of the form (1).

11. Linear combinations of rotation equations. As it was just shown in the last section that only linear combinations of (20) are transformable into rotation equations, here we shall transform equations (20) into their equivalent forms, by the matrix $\|t_{ij}\|$ ($i, j=1, 2, 3$), namely:

$$(25) \quad \begin{cases} t_{11}(u'_{y'}-v'_{x'})+t_{12}(u'_{z'}-w'_{x'})+t_{13}(v'_{z'}-w'_{y'})=0 \\ t_{21}(u'_{y'}-v'_{x'})+t_{22}(u'_{z'}-w'_{x'})+t_{23}(v'_{z'}-w'_{y'})=0 \\ t_{31}(u'_{y'}-v'_{x'})+t_{32}(u'_{z'}-w'_{x'})+t_{33}(v'_{z'}-w'_{y'})=0 \end{cases}$$

Again, it is equations (25) that we wish to transform into a new set of rotation equations. In doing so we are ready to apply our method of transformation to the new functions u' , v' , w' and the new variables x' , y' , z' (relative to u , v , w , x , y , z) by the following formulas

$$\begin{aligned}
 (26.1) \quad & \begin{cases} U = p_{11}u' + p_{12}v' + p_{13}w' \\ V = p_{21}u' + p_{22}v' + p_{23}w' \\ W = p_{31}u' + p_{32}v' + p_{33}w' \end{cases} & (26.2) \quad & \begin{cases} u' = p'_{11}U + p'_{12}V + p'_{13}W \\ v' = p'_{21}U + p'_{22}V + p'_{23}W \\ w' = p'_{31}U + p'_{32}V + p'_{33}W \end{cases} \\
 (26.3) \quad & \begin{cases} x' = q_{11}X + q_{12}Y + q_{13}Z \\ y' = q_{21}X + q_{22}Y + q_{23}Z \\ z' = q_{31}X + q_{32}Y + q_{33}Z \end{cases} & (26.4) \quad & \begin{cases} X = q'_{11}x' + q'_{12}y' + q'_{13}z \\ Y = q'_{21}x' + q'_{22}y' + q'_{23}z \\ Z = q'_{31}x' + q'_{32}y' + q'_{33}z \end{cases}
 \end{aligned}$$

Then, equations (25) can be brought to the following rotation equations,

$$(27) \quad \begin{cases} U_Y - V_X = 0 \\ U_Z - W_X = 0 \\ V_Z - W_Y = 0 \end{cases}$$

if, and only if, the following 27 equations are satisfied [cf. Section 1, especially (4), (5) and (6)]:

$$\begin{aligned}
 (28) \quad & \begin{cases} p_{11}q_{12} - p_{21}q_{11} = 0 \\ p_{11}q_{13} - p_{31}q_{11} = 0 \\ p_{21}q_{13} - p_{31}q_{12} = 0 \end{cases} & (33) \quad & \begin{cases} p_{12}q_{32} - p_{22}q_{31} = t_{13} \\ p_{12}q_{33} - p_{32}q_{31} = t_{23} \\ p_{22}q_{33} - p_{32}q_{32} = t_{33} \end{cases} \\
 (29) \quad & \begin{cases} p_{11}q_{22} - p_{21}q_{21} = t_{11} \\ p_{11}q_{23} - p_{31}q_{21} = t_{21} \\ p_{21}q_{23} - p_{31}q_{22} = t_{31} \end{cases} & (34) \quad & \begin{cases} p_{13}q_{12} - p_{23}q_{11} = -t_{12} \\ p_{13}q_{13} - p_{33}q_{11} = -t_{22} \\ p_{23}q_{13} - p_{33}q_{12} = -t_{32} \end{cases} \\
 (30) \quad & \begin{cases} p_{11}q_{32} - p_{21}q_{31} = t_{12} \\ p_{11}q_{33} - p_{31}q_{31} = t_{22} \\ p_{21}q_{33} - p_{31}q_{32} = t_{32} \end{cases} & (35) \quad & \begin{cases} p_{13}q_{22} - p_{23}q_{21} = -t_{13} \\ p_{13}q_{23} - p_{33}q_{21} = -t_{23} \\ p_{23}q_{23} - p_{33}q_{22} = -t_{33} \end{cases} \\
 (31) \quad & \begin{cases} p_{12}q_{12} - p_{22}q_{11} = -t_{11} \\ p_{12}q_{13} - p_{32}q_{11} = -t_{21} \\ p_{22}q_{13} - p_{32}q_{12} = -t_{31} \end{cases} & (36) \quad & \begin{cases} p_{13}q_{32} - p_{23}q_{31} = 0 \\ p_{13}q_{33} - p_{33}q_{31} = 0 \\ p_{23}q_{33} - p_{33}q_{32} = 0 \end{cases} \\
 (32) \quad & \begin{cases} p_{12}q_{22} - p_{22}q_{21} = 0 \\ p_{12}q_{23} - p_{32}q_{21} = 0 \\ p_{22}q_{23} - p_{32}q_{22} = 0 \end{cases}
 \end{aligned}$$

From systems of equations (28), (32) and (36) follow,

$$(37) \quad \begin{cases} \frac{p_{11}}{q_{11}} = \frac{p_{21}}{q_{12}} = \frac{p_{31}}{q_{13}} = \alpha \\ \frac{p_{12}}{q_{21}} = \frac{p_{22}}{q_{22}} = \frac{p_{32}}{q_{23}} = \beta \\ \frac{p_{13}}{q_{31}} = \frac{p_{23}}{q_{32}} = \frac{p_{33}}{q_{33}} = \gamma \end{cases}$$

Eliminating t_{ij} from (29) and (31), from (30) and (34), from (33) and (35) we get

$$(38) \quad \begin{cases} p_{11}q_{22} - p_{21}q_{21} + p_{12}q_{12} - p_{22}q_{11} = 0 \\ p_{11}q_{23} - p_{31}q_{21} + p_{12}q_{13} - p_{32}q_{11} = 0 \\ p_{21}q_{23} - p_{31}q_{22} + p_{22}q_{13} - p_{32}q_{12} = 0 \\ p_{11}q_{32} - p_{21}q_{31} + p_{13}q_{12} - p_{23}q_{11} = 0 \\ p_{11}q_{33} - p_{31}q_{31} + p_{13}q_{13} - p_{33}q_{11} = 0 \\ p_{21}q_{33} - p_{31}q_{32} + p_{23}q_{13} - p_{33}q_{12} = 0 \\ p_{12}q_{32} - p_{22}q_{31} + p_{13}q_{22} - p_{23}q_{21} = 0 \\ p_{12}q_{33} - p_{32}q_{31} + p_{13}q_{23} - p_{33}q_{21} = 0 \\ p_{22}q_{33} - p_{32}q_{32} + p_{23}q_{23} - p_{33}q_{22} = 0 \end{cases}$$

If the conditions in (37) and (38) are satisfied then it is possible to find a transformation matrix $\| t_{ij} \|$ and transformation formulas (26), that will bring equations (20) into (27). Our present problem is, therefore, to find p_{ij} and q_{ij} that satisfy (37) and (38) or conditions under which they do satisfy.

Substituting p_{ij} in terms of q_{ij} as given by (37) in (38) and simplifying, we have

$$(39) \quad \begin{cases} (\alpha - \beta) (q_{22}q_{11} - q_{12}q_{21}) = 0 \\ (\alpha - \beta) (q_{11}q_{23} - q_{13}q_{21}) = 0 \\ (\alpha - \beta) (q_{12}q_{23} - q_{13}q_{22}) = 0 \\ (\alpha - \gamma) (q_{11}q_{32} - q_{12}q_{31}) = 0 \\ (\alpha - \gamma) (q_{11}q_{33} - q_{13}q_{31}) = 0 \\ (\alpha - \gamma) (q_{12}q_{33} - q_{13}q_{32}) = 0 \\ (\beta - \gamma) (q_{21}q_{32} - q_{22}q_{31}) = 0 \\ (\beta - \gamma) (q_{21}q_{33} - q_{23}q_{31}) = 0 \\ (\beta - \gamma) (q_{22}q_{33} - q_{23}q_{32}) = 0 \end{cases}$$

A solution of (39) is, therefore, either

$$(40) \begin{cases} q_{22}q_{11}-q_{12}q_{21}=0 \\ q_{23}q_{11}-q_{13}q_{21}=0 \\ q_{12}q_{23}-q_{13}q_{22}=0 \end{cases} \quad (41) \begin{cases} q_{11}q_{32}-q_{12}q_{31}=0 \\ q_{11}q_{33}-q_{13}q_{31}=0 \\ q_{12}q_{33}-q_{13}q_{32}=0 \end{cases}$$

$$(42) \begin{cases} q_{21}q_{32}-q_{22}q_{31}=0 \\ q_{21}q_{33}-q_{23}q_{31}=0 \\ q_{22}q_{33}-q_{23}q_{32}=0 \end{cases}$$

or,

$$(43) \quad \alpha-\beta=0, \quad \alpha-\gamma=0, \quad \beta-\gamma=0$$

or, some among (40), (41) and (42) with some in (43) combined. But none of (40), (41) and (42) can be true for, if it were it would imply

$$(44) \quad \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} = 0$$

which contradicts our hypothesis of inverse transformability in (26.3). Consequently, equations (43) have to be true to preserve (39), therefore, equations (37) can be written thus

$$(45) \quad \frac{p_{ij}}{q_{ji}} = \alpha \quad (i, j = 1, 2, 3)$$

With the simple restriction (45) on our introduced constants p_{ij} and q_{ij} we are enabled to transform rotation equations (20) into (27), that is, into themselves. So we have complete freedom to choose q_{ij} to be whatever we want if only p_{ij} are kept for the purpose of satisfying condition (45). We will make use of our freedom over q_{ij} .

12. Transformation into divergence equation. By the transformation formulas (26) and the additional condition (45) we wish to transform equation (21) into a divergence equation. As before (cf. Section 2), by partial differentiation under change of variables and functions according to (26.1) and (26.3), the equation (21) can be transformed into

$$(49) \quad U_x + V_y + W_z = 0$$

when, and only when, the following nine equations hold, which are obtained from identifying coefficients of (21) with those of another equation derived from (46) by means of (26.1) and (26.3),

$$(47) \quad \begin{cases} p_{11}q_{11} + p_{21}q_{12} + p_{31}q_{13} = E_{11} \\ p_{11}q_{21} + p_{21}q_{22} + p_{31}q_{23} = E_{12} \\ p_{11}q_{31} + p_{21}q_{32} + p_{31}q_{33} = E_{13} \\ p_{12}q_{11} + p_{22}q_{12} + p_{32}q_{13} = E_{21} \\ p_{12}q_{21} + p_{22}q_{22} + p_{32}q_{23} = E_{22} \\ p_{12}q_{31} + p_{22}q_{32} + p_{32}q_{33} = E_{23} \\ p_{13}q_{11} + p_{23}q_{12} + p_{33}q_{13} = E_{31} \\ p_{13}q_{21} + p_{23}q_{22} + p_{33}q_{23} = E_{32} \\ p_{13}q_{31} + p_{23}q_{32} + p_{33}q_{33} = E_{33} \end{cases}$$

In accordance with the condition (45) equations (47) may be written

$$(48.1) \quad q_{11}q_{11} + q_{12}q_{12} + q_{13}q_{13} = \frac{E_{11}}{\alpha}$$

$$(48.2) \quad q_{11}q_{21} + q_{12}q_{22} + q_{13}q_{23} = \frac{E_{12}}{\alpha}$$

$$(48.3) \quad q_{11}q_{31} + q_{12}q_{32} + q_{13}q_{33} = \frac{E_{13}}{\alpha}$$

$$(48.4) \quad q_{21}q_{11} + q_{22}q_{12} + q_{23}q_{13} = \frac{E_{21}}{\alpha}$$

$$(48.5) \quad q_{21}q_{21} + q_{22}q_{22} + q_{23}q_{23} = \frac{E_{22}}{\alpha}$$

$$(48.6) \quad q_{21}q_{31} + q_{22}q_{32} + q_{23}q_{33} = \frac{E_{23}}{\alpha}$$

$$(48.7) \quad q_{31}q_{11} + q_{32}q_{12} + q_{33}q_{13} = \frac{E_{31}}{\alpha}$$

$$(48.8) \quad q_{31}q_{21} + q_{32}q_{22} + q_{33}q_{23} = \frac{E_{32}}{\alpha}$$

$$(48.9) \quad q_{31}q_{31} + q_{32}q_{32} + q_{33}q_{33} = \frac{E_{33}}{\alpha}$$

Conditions (48) are necessary and sufficient for a successful transformation of (21) into a divergence equation (46) without destroying the already transformed rotation equations (27). Our next task is to solve equations (48) for q_{ij} and α . First of all we observe that the left-hand members of (48.1), (48.5) and (48.9) are sums of squares, therefore, a solution is possible only when

$$(49) \quad \frac{E_{11}}{\alpha} > 0, \quad \frac{E_{22}}{\alpha} > 0, \quad \frac{E_{33}}{\alpha} > 0$$

Since we can choose α to be of the same sign as E_{11} the inequalities in (49) are reduced to

$$(50) \quad E_{11}E_{22} > 0, \quad E_{11}E_{33} > 0$$

which means that E_{11} , E_{22} and E_{33} must be of the same sign.

Furthermore, the left-hand members of (48.2) and (48.4) are identical and so are those of (48.3) and (48.7), of (48.6) and (48.8), it is obviously necessary to have

$$(51) \quad E_{12} = E_{21}, \quad E_{13} = E_{31}, \quad E_{23} = E_{32}$$

But, as we saw in (23), E_{ij} depend on d_{11} , d_{12} , d_{13} and d_{14} which are arbitrary. So we may take this opportunity to choose our d_{11} , d_{12} , d_{13} and d_{14} such that condition (51) will be satisfied.

According to (21) our E_{ij} may be written as follows, after collecting terms of d_{1k} ($k=1,2,3,4$),

$$\begin{cases} E_{12} = d_{11}I_{12} + d_{12}J_{12} + d_{13}K_{12} + d_{14}L_{12} \\ E_{21} = d_{11}I_{21} + d_{12}J_{21} + d_{13}K_{21} + d_{14}L_{21} \\ E_{31} = d_{11}I_{31} + d_{12}J_{31} + d_{13}K_{31} + d_{14}L_{31} \\ E_{13} = d_{11}I_{13} + d_{12}J_{13} + d_{13}K_{13} + d_{14}L_{13} \\ E_{23} = d_{11}I_{23} + d_{12}J_{23} + d_{13}K_{23} + d_{14}L_{23} \\ E_{32} = d_{11}I_{32} + d_{12}J_{32} + d_{13}K_{32} + d_{14}L_{32} \end{cases}$$

where I , J , K and L stand for functions of A_{ij} , B_{ij} , C_{ij} , D_{ij} , a'_{ij} and b'_{ij} while their subscripts indicate to which E_{ij} they belong. Equating E_{ij} from the above equations as demanded by (51) we get

$$\begin{cases} d_{11}(I_{12}-I_{21}) + d_{12}(J_{12}-J_{21}) + d_{13}(K_{12}-K_{21}) + (L_{12}-L_{21}) = 0 \\ d_{11}(I_{13}-I_{31}) + d_{12}(J_{13}-J_{31}) + d_{13}(K_{13}-K_{31}) + (L_{13}-L_{31}) = 0 \\ d_{11}(I_{23}-I_{32}) + d_{12}(J_{23}-J_{32}) + d_{13}(K_{23}-K_{32}) + (L_{23}-L_{32}) = 0 \end{cases}$$

which can always be satisfied by choosing d_{11} , d_{12} , d_{13} and d_{14} . Thus, equations (51) are no longer a condition.

13. Application of vector method to solve nine non-linear equations. With inequalities (50) as necessary conditions we now proceed to solve equations (48) whose left-hand quantities appear as scalar products of vectors. So we are led to invent three vectors

$$(52) \quad \begin{cases} Q_1 = q_{11}i + q_{12}j + q_{13}k \\ Q_2 = q_{21}i + q_{22}j + q_{23}k \\ Q_3 = q_{31}i + q_{32}j + q_{33}k \end{cases}$$

whose scalar products are

$$(53) \quad \begin{cases} Q_1^2 = q_{11}^2 \beta = \frac{E_{11}}{\alpha} & Q_1 Q_2 = q_{11} q_{21} \beta = \frac{E_{12}}{\alpha} \\ Q_2^2 = q_{21}^2 \beta = \frac{E_{22}}{\alpha} & Q_1 Q_3 = q_{11} q_{31} \beta = \frac{E_{13}}{\alpha} \\ Q_3^2 = q_{31}^2 \beta = \frac{E_{33}}{\alpha} & Q_2 Q_3 = q_{21} q_{31} \beta = \frac{E_{23}}{\alpha} \end{cases}$$

($\beta = \sum 1, 2, 3$). Now, conversely, if six numbers, that is, six scalar products $Q_i Q_j$ ($i, j = 1, 2, 3$) are given then can we find the components q_{ij} of three vectors so as to satisfy (53)? The answer is affirmative.

Since the length of the vector Q_1 is given by (53.1) we can fix our X-axis along Q_1 which can be expressed in co-ordinate system as

$$(54) \quad Q_1 = q_{11}i \quad (q_{12} = q_{13} = 0)$$

or, $q_{11} = |Q_1|$

with the given length of a vector Q_2 by (53.2) we can choose it to be lying in the X,Z-plane making an angle with the X-axis satisfying the following scalar product,

$$(55) \quad Q_1 Q_2 = Q_1 \cdot Q_2 \cos(Q_1, Q_2)$$

where,

$$(56) \quad Q_2 = q_{21}i + q_{22}j + q_{23}k \quad (q_{22} = 0).$$

Hence, (55) becomes

$$(57) \quad Q_1 Q_2 = q_{11}q_{21} \quad (q_{12} = q_{13} = 0)$$

$$\text{or,} \quad q_{21} = \frac{Q_1 Q_2}{|Q_1|}$$

In like manner a vector Q_3 with given length by (53.3) can be found making proper angles with Q_1 and Q_2 so as to satisfy its scalar products:

$$(58) \quad Q_1 Q_2 = q_{11}q_{31} \quad (q_{12} = q_{13} = 0)$$

$$\text{or,} \quad q_{31} = \frac{Q_1 Q_3}{|Q_1|}$$

$$(59) \quad Q_2 Q_3 = q_{21}q_{31} + q_{23}q_{33} \quad (q_{22} = 0).$$

By squaring in (56) we get

$$(60) \quad q_{23}^2 = Q_2^2 - q_{21}^2 = \frac{Q_1^2 Q_2^2 - (Q_1 Q_2)^2}{Q_1^2}$$

Substitute the value of q_{23} from (60) in (59) we obtain

$$(61) \quad q_{33} = \frac{Q_1^2 (Q_2 Q_3) - (Q_1 Q_2) (Q_1 Q_3)}{\sqrt{Q_1^2} \sqrt{Q_1^2 Q_2^2 - (Q_1 Q_2)^2}}$$

And lastly, q_{32} can be found from

$$Q_3^2 = q_{31}^2 + q_{32}^2 + q_{33}^2$$

where all other quantities are known.

For reference and clearness we shall tabulate q_{ij} in terms of E_{ij} below: (62)

$$q_{11} = \sqrt{\frac{E_{11}}{\alpha}}, \quad q_{12} = 0,$$

$$q_{13} = 0$$

$$q_{21} = \frac{E_{12}}{\sqrt{\alpha E_{11}}}, \quad q_{22} = 0,$$

$$q_{23} = \sqrt{\frac{E_{11} E_{22} - E_{12}^2}{\alpha E_{11}}}$$

$$q_{31} = \frac{E_{13}}{\sqrt{\alpha E_{11}}}, \quad q_{32} = \sqrt{\frac{E_{33}}{\alpha} - \frac{E_{13}^2}{\alpha E_{11}} - \frac{(E_{11}E_{23} - E_{12}E_{13})^2}{\alpha E_{11}(E_{11}E_{22} - E_{12}^2)}},$$

$$q_{33} = \frac{E_{11}E_{23} - E_{12}E_{13}}{\sqrt{\alpha E_{11}(E_{11}E_{22} - E_{12}^2)}}$$

In the above equations (62) all q_{ij} will be real if the following inequalities hold.

$$(63) \quad \begin{cases} E_{11}E_{22} - E_{12}^2 > 0 \\ (E_{11}E_{33} - E_{13}^2)(E_{11}E_{22} - E_{12}^2) - (E_{11}E_{23} - E_{12}E_{13})^2 \end{cases}$$

$$= E_{11} \begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{vmatrix}$$

Turning back to inequalities (50) we see that $E_{11} E_{22} > 0$ is a necessary condition for (63.1) to be true and that $E_{11} E_{33} > 0$ is a necessary condition for (63.2) to be true. Therefore, the conditions in (50) are all contained in (63).

Thus, the divergence problem is solved since we have found a necessary and sufficient condition in (63) under which we can find a transformation matrix $\|t_{ij}\|$ ($i, j=1, 2, 3$) and transformation formulas (26) that will carry over equations (20) and (21) into Newtonian equations of (27) and (46).

14. A summary of this chapter. We started our problem in this chapter with four given equations (2) and wanted to transform them into Newtonian equations of the form (1).

In Part First we succeeded in finding ten conditions (8) and (12), necessary and sufficient, on three of four given equations for their transformability into rotation equations by the method of transforming functions and variables.

On combining three given equations linearly we failed to get anything new; so in Part Second we transformed four given equations and reduced the previous ten conditions to seven, namely, (18) and (19) which are necessary and sufficient for transformability of four given equations into four new ones three of which have to be rotation equations.

Then in Part Third we undertook to transform the previous four "new" (transformed) equations into Newtonian equations which we did under a necessary and sufficient condition of inequalities in (63).

To sum up our solution of the problem in this chapter, we have found nine conditions in (18), (19) and (63), necessary and sufficient, under which four given linear homogeneous differential equations like (2) can be transformed into Newtonian equations of the form (1).

The writer wishes to express his thanks to the members of the Mathematics Faculty with whom he has studied, and especially to Professor Rainich under whose supervision this paper was written and to whose helpful and kindly interest he is greatly indebted.

